

# Modeling and Simulation of Bivariate Gaussian Random Fields

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**Abstract:** Multivariate data measured in space, such as temperature and pressure or the content of two metals in geological deposits, requires models that allow to incorporate spatial and cross-dependence of observations. We introduce some novel bivariate models, the powered exponential (or stable) covariance model and the Cauchy covariance model with flexible smoothness, variance, scale and cross-correlation parameters. In addition, we present a bundle of compactly supported bivariate covariance models obtained by the cut-off embedding technique from covariance functions with Euclid's hat scale mixture representation. Finally, we show that the circulant embedding algorithm always works if all covariance components have compact support. On the way we extend the circulant embedding method even for univariate models.

**Keywords and phrases:** matrix-valued covariance function, cross-correlation, multivariate Gaussian random fields, multivariate geostatistics, bivariate powered exponential model, bivariate stable model, bivariate Cauchy model, bivariate Matérn model.

## 1. Introduction

Weather forecasts, stemming from numerical weather prediction models, are highly sensitive to initial conditions and therefore are subject to bias. That is why multiple forecasts are produced by making small alterations either to the starting conditions or to the forecast model itself, or both. Additionally, this ensemble forecast is commonly postprocessed using statistical techniques to improve the calibration and to correct for potential biases. It is important that a postprocessing technique takes into account the spatial dependence of the observed phenomena. To this end an ensemble forecast can be corrected by a Gaussian error field, see for example [12, 2, 14] and the references therein. Spatial meteorological data generally have several components, for example temperature and pressure, and flexible multivariate models are required to reflect both spatial correlation and cross-correlation adequately. A flexible multivariate model should allow different degrees of smoothness, scale, variance and cross-correlation of its components. Such models are suggested in [25, 19, 8, 29], generalizing well-known univariate covariance models to multivariate ones. In this paper we introduce a bivariate Cauchy model and a bivariate powered exponential (or bivariate stable) model. Both models are flexible, intuitive and easily interpretable: in both models three parameters characterize the smoothness of covariances of process components and cross-covariance, and further three parameters are responsible for long-range behaviour of covariances in the bivariate Cauchy model.

Circulant embedding is a powerful tool for simulating Gaussian random fields on a rectangular grid in  $\mathbb{R}^n$ . It was introduced for univariate fields by [32] and [10] and was extended to the multivariate case by [6] and [21]. The algorithm is based on embedding the covariance matrix into a larger matrix with circulant structure or block circulant structure. For several covariance functions with non-compact support a classical circulant embedding algorithm is either inefficient or inapplicable at all (especially in dimensions higher than 2), since the circulant embedding matrix is not positive definite. A circumvention of this difficulty is the cut-off circulant embedding technique introduced by [20, 31].

No easy-to-use method for simulating multivariate fields allowing for fast and exact simulation exists to the best of our knowledge. Moreover, it is very difficult to check if the covariance matrix has a nonnegative circulant embedding matrix via the conditions given in [6]. So, we will prove that if all the components of a matrix-valued covariance function have compact support, the corresponding

covariance matrix has a nonnegative definite circulant embedding matrix. Models introduced here and for example, in [19] do not have compact supports and therefore we generalize the univariate cut-off embedding technique to the bivariate case.

The paper is organized as follows. Section 2 recalls some basic notions and facts about covariance functions. In Section 3 we review existing covariance models for multivariate Gaussian random fields. Section 4 introduces a new class of bivariate covariance functions based on Euclid's hat scale mixture representation. Bivariate Cauchy model and bivariate stable model are members of this class. Section 5 extends the cut-off embedding algorithm for a class of univariate covariance function not included in [20] and uses this extension to introduce the bivariate version of the simulation method.

## 2. Notation and Basic Facts

We consider a centered  $m$ -variate Gaussian random field  $Z$  in  $\mathbb{R}^n$

$$Z(x) = (Z_1(x), \dots, Z_m(x)), \quad x \in \mathbb{R}^n.$$

Such a random field is uniquely characterized by its cross-covariance  $C$ , a matrix-valued function, where each diagonal element  $C_{ii}(x, y) = \mathbb{E}Z_i(x)Z_i(y)$ ,  $x, y \in \mathbb{R}^n$ , is a univariate covariance function. Each off-diagonal element  $C_{ij}(x, y) = \mathbb{E}Z_i(x)Z_j(y)$  is the cross-covariance between process components  $1 \leq i \neq j \leq m$ . As in a univariate case a square integrable random field  $Z$  is called (weakly) stationary, if its covariance function  $C$  is translation invariant, and isotropic, if for all  $x, y \in \mathbb{R}^n$  and rotation matrices  $A \in \mathbb{R}^{n \times n}$ ,

$$C(x, y) = C(Ax, Ay).$$

In this paper we consider centered stationary and isotropic Gaussian random fields, thus their covariance depends only on the distance  $r = \|x - y\|$  between locations  $x$  and  $y$  and there exist functions  $\varphi_{ij} : [0, \infty) \mapsto \mathbb{R}$  such that  $C_{ij}(x, y) = \varphi_{ij}(r)$ ,  $i, j = 1, \dots, m$ . Necessarily,  $\varphi_{ij} = \varphi_{ji}$  for all  $i, j = 1, \dots, m$ .

We denote by  $\Phi_n$  the set of continuous functions  $\varphi : [0, \infty) \mapsto \mathbb{R}$  such that the map  $(x, y) \mapsto \varphi(\|x - y\|)$  is positive definite in  $\mathbb{R}^n$ . Similarly,  $\Phi_n^m$  denotes the class of mappings  $\varphi = [\varphi_{ij}(\cdot)]_{i,j=1}^m : [0, \infty) \mapsto \mathbb{R}^{m \times m}$  with each  $\varphi_{ij}$  being continuous, such that

$$C(x, y) = [\varphi_{ij}(\|x - y\|)]_{i,j=1}^m, \quad x, y \in \mathbb{R}^n,$$

is an  $m \times m$  matrix-valued covariance function in  $\mathbb{R}^n$ . We recall the definition of a positive definite matrix-valued function.

**Definition 2.1.** An  $m \times m$ -matrix-valued function  $C : \mathbb{R}^n \mapsto \mathbb{R}^{m \times m}$  is called positive definite, if for any  $p \in \mathbb{N}$  and  $a_1, \dots, a_p \in \mathbb{R}^m$ ,  $x_1, \dots, x_p \in \mathbb{R}^n$

$$\sum_{i=1}^p \sum_{j=1}^p a_i^T C(x_i - x_j) a_j \geq 0.$$

The following result is the multidimensional generalization of Cramer's theorem [7], which is itself a multivariate generalization of Bochner's theorem [4] and can be found for example in [16].

**Theorem 2.1.** A translation invariant  $m \times m$ -matrix-valued function  $C$  is continuous and positive definite if and only if there exists a matrix-valued finite measure  $F$ , such that  $F(A)$  is positive definite matrix for all  $A \in \mathcal{B}^n$  and

$$C(x) = \int_{\mathbb{R}^n} e^{i\langle x, h \rangle} dF(h) \quad \text{for all } x \in \mathbb{R}^n.$$

If the  $C_{ij}$  are additionally absolutely integrable, then it holds that

$$C(x) = \int_{\mathbb{R}^n} e^{i\langle x, h \rangle} f(h) dh \quad \text{for all } x \in \mathbb{R}^n \quad (2.1)$$

with an  $m \times m$ -matrix-valued function  $f(h)$ , which is a positive semi-definite matrix for all  $h \in \mathbb{R}^n$ , see [33].

The isotropy of  $C$  leads to the isotropy of  $f$ , i.e.  $f(h) = f(\|h\|)$  and with  $u = \|h\|$  and  $C(x) = \varphi(\|x\|)$  equation (2.1) yields the Hankel type transform of  $f$

$$\varphi(r) = (2\pi)^{n/2} \int_0^\infty (ru)^{-(n-2)/2} J_{(n-2)/2}(ru) u^{n-1} f(u) du, \quad r \geq 0,$$

or, more generally

$$\varphi(r) = 2^{(n-2)/2} \Gamma(n/2) \int_0^\infty (ru)^{-(n-2)/2} J_{(n-2)/2}(ru) F(du), \quad r \geq 0,$$

where  $J_\nu$  is the Bessel function of the first kind, see for example [30, 22].

We quote the Tauberian theorem from [3, 22] which links the properties of a measure with those of its Hankel type transform. We will use this link in Section 4 to explain why the smoothness parameter of the cross-covariance in the bivariate stable model and both the smoothness parameter and long-range parameter of the cross-covariance in the bivariate Cauchy model are bounded from below. We first need the notion of regular variation.

**Definition 2.2.** A function  $L : (0, \infty) \mapsto [0, \infty)$  is said to be regularly varying at infinity with index  $\alpha$ , if for every  $\lambda > 0$ ,

$$\frac{L(\lambda r)}{L(r)} \rightarrow r^\alpha \text{ as } r \rightarrow \infty.$$

If  $\alpha = 0$ , the function  $L$  is said to be slowly varying at infinity.

**Theorem 2.2** (Tauberian theorem). *Let  $F$  be a probability measure on  $[0, \infty)$ ,  $\varphi$  be its Hankel type transform in  $\mathbb{R}^n$  and  $L$  a function varying slowly at infinity.*

- If  $0 < \alpha < 2$ , then

$$1 - \varphi(r) \sim r^\alpha L(1/r) \text{ as } r \rightarrow 0+ \quad (2.2)$$

if and only if

$$1 - F(u) \sim \frac{L(u)}{u^\alpha} \frac{2^\alpha \Gamma(n/2 + \alpha/2)}{\Gamma(n/2) \Gamma(1 - \alpha/2)} \text{ as } u \rightarrow \infty. \quad (2.3)$$

If  $\alpha = 2$ , relation (2.2) is equivalent to

$$\int_0^r u[1 - F(u)] du \sim nL(r) \text{ as } r \rightarrow \infty$$

or to

$$\int_0^r u^2 F(du) \sim 2nL(r) \text{ as } r \rightarrow \infty.$$

If  $\alpha = 0$ , the relation (2.2) implies the asymptotic equivalence (2.3). Conversely, (2.3) implies (2.2) with  $\alpha = 0$  if  $[1 - F(u)]$  is convex for  $u$  sufficiently large, but not in general.

- Let  $0 < \beta < n$ . If

$$\varphi(r)r^\beta \sim c(n, \beta)L(r) \text{ as } r \rightarrow \infty,$$

where  $c(n, \beta) = \frac{2^\beta \Gamma(\frac{\beta+2}{2}) \Gamma(\frac{n}{2})}{\Gamma(\frac{n-\beta}{2})}$ , then

$$F(u)/u^\beta \sim L\left(\frac{1}{u}\right) \text{ as } u \rightarrow 0+.$$

- Let  $\beta > n$ , then

$$\varphi(r)r^\beta \sim \text{const } L(r) \text{ as } r \rightarrow \infty,$$

if and only if

$$F(u)/u^\beta \sim \text{const } L(u) \text{ as } u \rightarrow 0+.$$

*Example 2.1* (Powered exponential covariance). Let  $\varphi(r) = e^{-r^\alpha}$ ,  $r > 0$ ,  $\alpha \in (0, 2)$ . Then  $1 - \varphi(r)$  is regularly varying with index  $\alpha$  at  $r = 0$ . As

$$\lim_{r \rightarrow 0} \frac{1 - e^{-r^\alpha}}{r^\alpha} = \lim_{r \rightarrow 0} \frac{\alpha e^{-r^\alpha} r^{\alpha-1}}{\alpha r^{\alpha-1}} = 1$$

we have  $L(r) = 1$ . The corresponding measure  $F$  varies at infinity as follows

$$1 - F(u) \sim u^{-\alpha} \frac{2^\alpha \Gamma((n + \alpha)/2)}{\Gamma(n/2) \Gamma(1 - \alpha/2)} \text{ as } u \rightarrow \infty.$$

Since the function  $\varphi$  decreases rapidly enough at infinity, the density  $f$  of  $F$  exists and

$$f(u) \sim \alpha u^{-\alpha-1} \frac{2^\alpha \Gamma((n + \alpha)/2)}{\Gamma(n/2) \Gamma(1 - \alpha/2)} \text{ as } u \rightarrow \infty.$$

The latter matches with series representation of  $f$  in [24].

*Example 2.2* (Cauchy type covariance). Let  $\varphi(r) = (1 + r^\alpha)^{-\beta/\alpha}$ ,  $r, \beta > 0$ ,  $\alpha \in (0, 2)$ . Then  $1 - \varphi(r)$  is regularly varying with index  $\alpha$  at  $r = 0$ ,

$$\lim_{r \rightarrow 0} \frac{1 - (1 + r^\alpha)^{-\beta/\alpha}}{r^\alpha} = \lim_{r \rightarrow 0} \frac{\beta(1 + r^\alpha)^{-\beta/\alpha-1} r^{\alpha-1}}{\alpha r^{\alpha-1}} = \frac{\beta}{\alpha},$$

and  $L(r) = \beta/\alpha$ . The density  $f$  of  $F$  decays at infinity as follows

$$f(u) \sim u^{-\alpha-1} \frac{2^\alpha \beta \Gamma((n + \alpha)/2)}{\Gamma(n/2) \Gamma(1 - \alpha/2)} \text{ as } u \rightarrow \infty.$$

This matches the series representation for the spectral density of the Cauchy covariance in [23]. Analogously, the density  $f$  behaves at the origin as

$$f(u) \sim \text{const } u^{\beta-1} \text{ as } u \rightarrow 0.$$

### 3. Matrix-valued Covariances Built from Univariate Models

We provide some examples of matrix-valued covariance functions built from univariate models. An overview of cross-covariance functions for multivariate geostatistics is found in [15] and [28]. For multivariate models based on normal scale mixtures see [27].

#### 3.1. Separable models

A cross-covariance matrix function is separable if

$$C_{ij}(x) = \rho(x) R_{ij}, \quad x \in \mathbb{R}^n,$$

for all  $i, j = 1, \dots, m$ , where  $\rho(x)$  is a correlation function and  $R_{ij}$  is the nonspatial covariance between the variables  $i$  and  $j$  [15]. It has a compact support if and only if  $R$  is identically zero or  $\rho$  has a compact support.

#### 3.2. Convolutions

Suppose that  $c_1, \dots, c_m$  are real-valued functions on  $\mathbb{R}^n$  which are both integrable and square-integrable. If

$$C_{ij}(x) = (c_i * c_j)(x) \quad \text{for } i, j = 1, \dots, m, \quad (3.1)$$

where the asterisk  $*$  denotes the convolution operator, the matrix-valued function defined by equation (3.1) is a multivariate covariance function on  $\mathbb{R}^n$  (Theorem 2 in [19]). The multivariate Matérn covariance model serves as an example of this construction. The function  $C$  has a compact support if  $c_i$  are all compactly supported.

### 3.3. Mixtures

We quote a theorem from [26].

**Theorem 3.1.** (A) Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space and  $E$  be a linear space. Assume that the family of matrix-valued functions  $A(x, u) = [A_{ij}(x, u)] : E \times \Omega \mapsto \mathbb{C}^{m \times m}$  satisfies the following conditions:

1. for every  $i, j = 1, \dots, m$  and  $x \in E$ , the functions  $A_{ij}(x, \cdot)$  belong to  $L_1(\Omega, \mathcal{F}, \mu)$ ;
2.  $A(\cdot, u)$  is a positive definite matrix-valued function for  $\mu$ -almost every  $u \in \Omega$ .

Let

$$C(x) := \int_{\Omega} A(x, u) d\mu(u) = \left[ \int_{\Omega} A_{ij}(x, u) d\mu(u) \right]_{i,j=1}^m, \quad x \in E.$$

Then  $C$  is a positive definite matrix-valued function.

(B) Conditions (1) and (2) are satisfied when  $A(x, u) = f(x, u)F(x, u)$ , where the maps  $f(x, u) : E \times \Omega \mapsto \mathbb{C}$  and  $F(x, u) = [F_{ij}(x, u)]_{i,j=1}^m : E \times \Omega \mapsto \mathbb{C}^{m \times m}$  satisfy the following conditions:

1. for every  $i, j = 1, \dots, m$  and  $x \in E$ , the functions  $f(x, \cdot)F_{ij}(x, \cdot)$  belong to  $L_1(\Omega, \mathcal{F}, \mu)$ ;
2.  $f(\cdot, u)$  is positive definite for  $\mu$ -almost every  $u \in \Omega$ ;
3.  $F(\cdot, u)$  is a positive definite matrix-valued function or  $F(\cdot, u) = F(u)$  is a positive definite matrix for  $\mu$ -almost every  $u \in \Omega$ .

Convolution and mixture approaches were used in [11] for constructing matrix-valued covariance functions with compact support. However, these models do not allow random field components to maintain distinct smoothness parameters.

#### 3.3.1. Euclid's Hat Scale Mixtures

We consider in particular a scale mixture of Euclid's hat functions  $h_n$ , i.e. the function  $\varphi \in \Phi_n$  with the representation

$$\varphi(r) = \int_0^\infty h_n(ru) dG(u), \quad r > 0, \quad (3.2)$$

where

$$h_n(r) = \begin{cases} \frac{n\Gamma(n/2)}{\pi^{1/2}\Gamma((n+1)/2)} \int_r^1 (1-v^2)^{(n-1)/2} dv, & r \leq 1, \\ 0, & r > 1, \end{cases}$$

defined on  $[0, \infty)$  and  $G(u)$  is a distribution function with  $G(0+) = 0$  [17]. The distribution  $G$  can be recovered from equation (3.2). Closed formulae are available for example for  $n = 1$ , where

$$h_1(r) = (1-r)_+, \quad G(r) = \varphi(1/r) - \frac{1}{r} \varphi'(1/r), \quad r \geq 0$$

and for  $n = 3$ , where

$$h_3(r) = (1-r)_+ - \frac{r}{2}(1-r^2)_+, \quad G(r) = \varphi(1/r) - \frac{1}{r} \varphi'(1/r) + \frac{1}{3r^2} \varphi''(1/r), \quad r \geq 0.$$

If  $G$  has a density  $g$  with respect to the Lebesgue measure, then  $g$  can be recovered for  $n = 1$  by

$$g(r) = \frac{1}{r^3} \varphi''\left(\frac{1}{r}\right) \quad (3.3)$$

and for  $n = 3$  by

$$g(r) = \frac{1}{3r^3} \left( \varphi''\left(\frac{1}{r}\right) - \frac{1}{r} \varphi''' \left(\frac{1}{r}\right) \right). \quad (3.4)$$

#### 4. Bivariate Fields with Euclid's Hat Scale Mixture Representation

We consider a bivariate field in  $\mathbb{R}^n$  with a covariance function  $\varphi$  given by

$$\varphi(r) = \begin{bmatrix} \sigma_{11}^2 \varphi_{11}(r) & \rho \sigma_{11} \sigma_{22} \varphi_{12}(r) \\ \rho \sigma_{11} \sigma_{22} \varphi_{12}(r) & \sigma_{22}^2 \varphi_{22}(r) \end{bmatrix}, \quad (4.1)$$

$\sigma_{11}, \sigma_{22} > 0, |\rho| \leq 1$  such that equation (4.1) defines a matrix-valued covariance function. We assume, that the  $\varphi_{ij}, i, j = 1, 2$ , are the scale mixtures of Euclid's hat function  $h_n$ , i.e.,

$$\varphi_{ij}(r) = \int_0^\infty h_n(ru) dG_{ij}(u),$$

where  $G_{ij}$  are distribution functions with  $G_{ij}(0+) = 0$  [17]. Additionally we assume that the  $G_{ij}$  have densities  $g_{ij}$ . Then, applying part B of Theorem 3.1 with  $f(r, u) = (1 - ru)_+$  in  $\mathbb{R}$  and  $f(r, u) = (1 - ru)_+ - \frac{ru}{2}(1 - (ru)^2)_+$  in  $\mathbb{R}^3$ ,  $F(r, u) = [g_{ij}(u)]_{i,j=1}^2$  and  $\mu$  the Lebesgue measure, we obtain that if

$$\begin{bmatrix} \sigma_{11}^2 g_{11}(r) & \rho \sigma_{11} \sigma_{22} g_{12}(r) \\ \rho \sigma_{11} \sigma_{22} g_{12}(r) & \sigma_{22}^2 g_{22}(r) \end{bmatrix},$$

is positive definite for almost all  $r \geq 0$ , then the function  $\varphi$  in equation (4.1) is a covariance function. This implies, that for the positive definiteness of the matrix (4.1), it is sufficient, that

$$\rho^2 \leq \inf_{r \geq 0} \frac{g_{11}(r)g_{22}(r)}{g_{12}^2(r)}.$$

We can formulate the following theorem.

**Theorem 4.1.** *A matrix-valued function  $C$  defined by equation (4.1), where each  $\varphi_{ij}$  admits the scale mixture representation with Euclid's hat, is positive definite*

- in  $\mathbb{R}$  if  $\varphi_{ij}$  is twice differentiable and

$$\rho^2 \leq \inf_{r \geq 0} \frac{\varphi_{11}''(r)\varphi_{22}''(r)}{(\varphi_{12}''(r))^2} \quad (4.2)$$

- in  $\mathbb{R}^3$  if  $\varphi_{ij}$  is three times differentiable and

$$\rho^2 \leq \inf_{r \geq 0} \frac{(\varphi_{11}''(r) - r\varphi_{11}'''(r))(\varphi_{22}''(r) - r\varphi_{22}'''(r))}{(\varphi_{12}''(r) - r\varphi_{12}'''(r))^2}. \quad (4.3)$$

*Proof.* Inequalities (4.2) and (4.3) follow from equations (3.3) and (3.4), respectively.  $\square$

##### 4.1. Bivariate Powered Exponential Model

The univariate powered exponential covariance model is defined for  $r \geq 0$  by

$$\varphi_{\alpha,a}(r) = e^{-(ar)^\alpha},$$

where  $a > 0$  is the scale parameter and the smoothness parameter  $\alpha$  lies in  $(0, 1]$ .

The following question is posed in [15]: how to characterise a parameter set of the valid multivariate powered exponential model? We give a partial answer to this question introducing a bivariate powered exponential covariance model  $\varphi$

$$\varphi(r) = \begin{bmatrix} \sigma_1^2 e^{-(ar)^\alpha} & \rho \sigma_1 \sigma_2 e^{-(br)^\beta} \\ \rho \sigma_1 \sigma_2 e^{-(br)^\beta} & \sigma_2^2 e^{-(cr)^\gamma} \end{bmatrix}, \quad (4.4)$$

where  $\sigma_1, \sigma_2, a, b, c > 0$ ,  $0 < \alpha, \beta, \gamma \leq 1$ ,  $|\rho| < 1$ .

We define the functions  $q_{\alpha,a}^{(n)}(r)$ ,  $n \in \{1, 3\}$  by

$$\begin{aligned} q_{\alpha,a}^{(1)}(r) &= \alpha(ar)^\alpha - \alpha + 1, \\ q_{\alpha,a}^{(3)}(r) &= \alpha^2(ar)^{2\alpha} - 3\alpha^2(ar)^\alpha + 4\alpha(ar)^\alpha + \alpha^2 - 4\alpha + 3. \end{aligned}$$

Using Theorem 4.1 we can show the following.

**Theorem 4.2.** *A matrix-valued function  $\varphi$  given by equation (4.4) is positive definite in  $\mathbb{R}^n$ ,  $n \in \{1, 3\}$  if*

$$\rho^2 \leq \inf_{r \geq 0} \frac{\alpha\gamma a^\alpha c^\gamma}{\beta^2 b^{2\beta}} r^{\alpha+\gamma-2\beta} e^{2(br)^\beta - (ar)^\alpha - (cr)^\gamma} \frac{q_{\alpha,a}^{(n)}(r)q_{\gamma,c}^{(n)}(r)}{(q_{\beta,b}^{(n)}(r))^2}, \quad (4.5)$$

In particular,

- (i) if  $\beta < \frac{\alpha+\gamma}{2}$  the model is valid only for  $\rho = 0$ ,
- (ii) if  $\beta = \alpha = \gamma$  the infimum (4.5) is positive if  $b^\alpha \geq \frac{a^\alpha + c^\alpha}{2}$  and 0 otherwise,
- (iii) if  $\beta = \alpha > \gamma$ , the infimum (4.5) is positive if  $b > 2^{-1/\alpha}a$  and 0 otherwise,
- (iv) if  $\beta = \gamma > \alpha$ , the infimum (4.5) is positive if  $b > 2^{-1/\gamma}c$  and 0 otherwise,
- (v) if  $\beta > \max\{\alpha, \gamma\}$  the infimum (4.5) is positive.

*Proof.* We proof the proposition for one-dimensional bivariate fields. The three-dimensional case is analogous.

(i) Example 2.1 and Cramer's Theorem imply that if the cross-covariance smoothness parameter  $\beta$  is smaller than the arithmetic mean of  $\alpha$  and  $\gamma$ , the matrix-valued function  $\varphi$  defined by equation (4.4) cannot be positive definite. Indeed, the matrix of spectral densities

$$\begin{bmatrix} \frac{1}{a}\sigma_1^2 f_\alpha(\frac{r}{a}) & \frac{1}{b}\rho\sigma_1\sigma_2 f_\beta(\frac{r}{b}) \\ \frac{1}{b}\rho\sigma_1\sigma_2 f_\beta(\frac{r}{b}) & \frac{1}{c}\sigma_2^2 f_\gamma(\frac{r}{c}) \end{bmatrix},$$

is not positive definite for some  $r > 0$ , since

$$\lim_{r \rightarrow \infty} \frac{b^2}{ac} \frac{f_\alpha(\frac{r}{a})f_\gamma(\frac{r}{c})}{f_\beta^2(\frac{r}{b})} = \lim_{r \rightarrow \infty} \frac{b^2}{ac} \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\sin(\frac{\pi\alpha}{2})\sin(\frac{\pi\gamma}{2})a^\alpha c^\gamma}{\Gamma(\beta+1)^2 \sin^2(\frac{\pi\beta}{2})b^{2\beta}} r^{2\beta-\alpha-\gamma} = 0.$$

(ii) - (v) The cases, where  $\beta \geq \frac{\alpha+\gamma}{2}$  are based on the derivatives of  $\varphi_{ij}$  plugged in into inequality (4.2) of Theorem 4.1. This leads directly to inequality (4.5). All factors of the right hand-side of inequality (4.5) are positive if  $r$  is positive. That means that the infimum can be zero only if  $r = 0$  or  $r = \infty$ . Clearly, for the parameter values given in (ii) - (v), the infima are strictly positive.  $\square$

*Remark 4.1.* Although for some parameter values the infimum in inequality (4.5) is 0, this does not imply that the matrix-valued function  $\varphi$  in equations (4.4) is not a covariance model. Inequalities (4.2) and (4.3) provide only a sufficient but not a necessary condition for positive definiteness. Thus, there is room for improving the  $\rho$  bounds.

Figure 1 provides an example of the infimum value in inequality (4.5) found numerically.

## 4.2. Bivariate Cauchy Model

We consider the Cauchy type univariate covariance model

$$\varphi_{\alpha,\lambda,a}(r) = (1 + (ar)^\alpha)^{-\lambda/\alpha},$$

where  $a > 0$  is the scale parameter,  $0 < \alpha \leq 1$  is the smoothness parameter and  $\lambda > 0$  models the long-range behaviour of the covariance function.

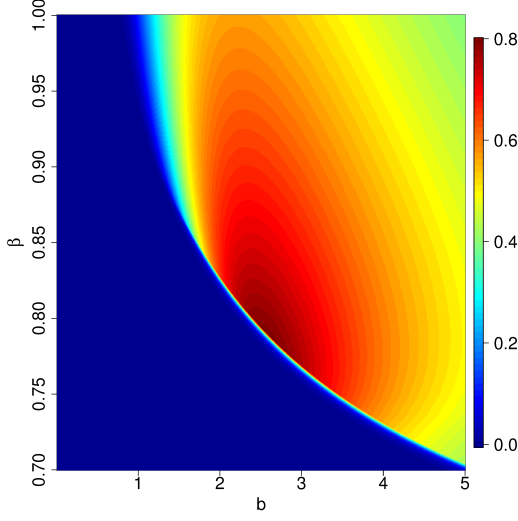


Fig 1: The maximum attainable  $|\rho|$  in inequality (4.5) for the bivariate powered exponential covariance model in  $\mathbb{R}$ . The parameters are:  $\alpha = 0.5, \gamma = 0.9, a = 2, c = 2.5$ .

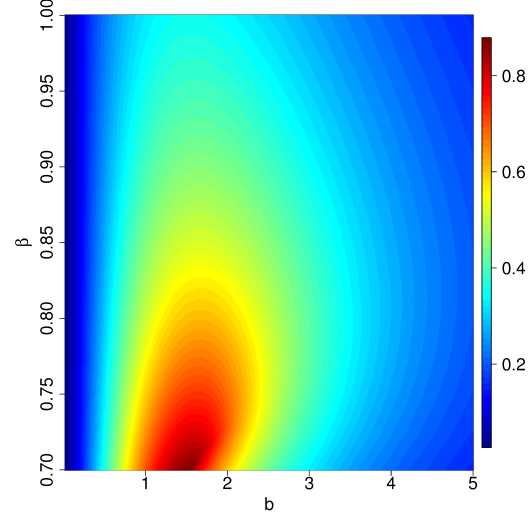


Fig 2: The maximum attainable  $|\rho|$  in inequality (4.7) for the bivariate Cauchy covariance model in  $\mathbb{R}$ . The parameters are  $\alpha = 0.5, \gamma = 0.9, \lambda = 2, \nu = 2.5, \mu = 2.1, a = 0.5, c = 0.7$ .

We introduce a bivariate Cauchy type covariance model  $\varphi$

$$\varphi(r) = \begin{bmatrix} \sigma_1^2(1 + (ar)^\alpha)^{-\lambda/\alpha} & \rho\sigma_1\sigma_2(1 + (br)^\beta)^{-\nu/\beta} \\ \rho\sigma_1\sigma_2(1 + (br)^\beta)^{-\nu/\beta} & \sigma_2^2(1 + (cr)^\gamma)^{-\mu/\gamma} \end{bmatrix}, \quad (4.6)$$

where  $\sigma_1, \sigma_2, a, b, c, \lambda, \mu, \nu > 0, 0 < \alpha, \beta, \gamma \leq 1, |\rho| < 1$ . We use again the scale mixture representation of  $\varphi$  and Theorem 4.1 to determine under which conditions the matrix (4.6) defines a covariance function.

We define the functions  $p_{\alpha, \lambda, a}^{(n)}(r), n \in \{1, 3\}$

$$p_{\alpha, \lambda, a}^{(1)}(r) = \frac{(\lambda + 1)(ar)^\alpha - \alpha + 1}{(1 + (ar)^\alpha)^{\lambda/\alpha + 2}}$$

$$p_{\alpha, \lambda, a}^{(3)}(r) = \frac{(\lambda + 1)(\lambda + 3)(ar)^{2\alpha} + (ar)^\alpha(4\lambda + 6 - 4\alpha - 3\lambda\alpha - \alpha^2) + (\alpha - 1)(\alpha - 3)}{(1 + (ar)^\alpha)^{\lambda/\alpha + 3}}$$

**Theorem 4.3.** A matrix-valued function  $\varphi$  given by equation (4.6) is positive definite in  $\mathbb{R}^n, n \in \{1, 3\}$  if

$$\rho^2 \leq \inf_{r \geq 0} \frac{\lambda\mu a^\alpha c^\gamma}{\nu^2 b^{2\beta}} r^{\alpha + \gamma - 2\beta} \frac{p_{\alpha, \lambda, a}^{(n)}(r)p_{\gamma, \mu, c}^{(n)}(r)}{(p_{\beta, \nu, b}^{(n)}(r))^2} \quad (4.7)$$

In particular,

- (i) if  $\beta < \frac{\alpha + \gamma}{2}$  the model is valid only for  $\rho = 0$ ,
- (ii) if  $\nu < \frac{\lambda + \mu}{2}$  and  $\lambda, \nu, \mu \neq n$  the model is valid only for  $\rho = 0$ ,
- (iii) if  $\beta \geq \frac{\alpha + \gamma}{2}$  and  $\nu \geq \frac{\lambda + \mu}{2}$ , the infimum in inequality (4.7) is positive.

*Proof.* The proof of the proposition for a one-dimensional bivariate field is similar to Theorem 4.2 and the three-dimensional case is analogous.



(i) Analogously to the bivariate powered exponential model, Example 2.2 and Cramer's Theorem imply that if the cross-covariance smoothness parameter  $\beta$  is smaller than the arithmetic mean of  $\alpha$  and  $\gamma$ , the matrix-valued function  $\varphi$  given by equation (4.6) cannot be positive definite.

(ii) Similarly, if  $\nu < \frac{\lambda+\mu}{2}$  the matrix of spectral densities is not positive definite for some values near the origin, see Example 2.2.

(iii) All factors of the right hand-side of inequality (4.7) are positive and continuous for  $r > 0$ . Then the infimum can achieve 0 only if  $r = 0$  or  $r = \infty$ . For  $\beta \geq \frac{\alpha+\gamma}{2}$  and  $\nu \geq \frac{\lambda+\mu}{2}$  this is clearly not the case.  $\square$

Figure 2 provides an example of the infimum value in inequality (4.7) calculated numerically. Note that the bounds for  $|\rho|$  can be improved, since Theorem 4.3 provides only a sufficient condition for the positive definiteness of model (4.6).

## 5. Cut-off Embedding for Bivariate Fields

The idea of the cut-off embedding technique for the simulation of univariate Gaussian random fields is the following. Instead of simulating a random field with a covariance function  $\varphi(r)$ ,  $r \in [0, 1]$  on a grid  $G = [0, 1/\sqrt{n}]^n$ , a field  $Y$  on a larger grid with a continuous covariance function  $\chi(r)$  of the form

$$\chi(r) = \begin{cases} \varphi(r), & 0 \leq r \leq 1, \\ \psi(r), & 1 \leq r \leq R, \\ 0, & r \geq R, \end{cases}$$

is simulated. Here, the radius  $R$  and the function  $\psi$  are chosen such that  $\chi$  belongs to the class  $\Phi_n$ . Since the function  $\chi(r)$  has compact support, the circulant matrix is positive definite, whenever the size of the simulation window is bigger than the support. Then the part of the random field  $Y$  that lies within  $G$  has the required covariance function  $\varphi$ .

In this section we first extend the cut-off circulant embedding method for univariate fields to a class of functions which is not covered by Theorems 1 and 2 in [20] and then extend it to bivariate fields. We will end up with the following kind of model

$$\chi_{ij}(r) = \begin{cases} C_{ij} + \varphi_{ij}(r), & 0 \leq r \leq 1, \\ \psi_{ij}(r), & 1 \leq r \leq R_{ij}, \\ 0, & r \geq R_{ij}. \end{cases}$$

Our goal is to choose the functions  $\psi_{ij}$ , the constants  $C_{ij}$  and  $R_{ij}$ ,  $i, j = 1, 2$  such that the modified matrix  $[\chi_{ij}]_{i,j=1}^2$  is a matrix-valued covariance function in  $\mathbb{R}^n$ . Then the following lemma ensures the existence of a nonnegative definite embedding for matrix  $\chi$ .

**Lemma 5.1.** *A stationary and isotropic matrix-valued covariance function, which elements are continuous, absolutely integrable, of bounded variation and have compact supports, allows a nonnegative definite circulant embedding if the diameter of the simulation window exceeds the highest diameter of the supports.*

*Proof.* The nonnegativity of the eigenvalues of the circulant embedded matrices follows directly from the Poisson summation formula, see [13, 5], and its extension to  $\mathbb{R}^n$ , see the end of Chapter 5 of [1]. The one-dimensional case is considered in Appendix B of [9]. In a multivariate extension of the circulant embedding algorithm, a so-called 'synthesis phase' arises. It is described in details for one-dimensional multivariate covariances in [21]. The  $n$ -dimensional generalizations follows straightforward by replacing the one-dimensional Fourier transform by the  $n$ -dimensional one.  $\square$

*Remark 5.1.* The requirement of bounded variation can be replaced with other regularity conditions. See Chapter 5 of [1] for a discussion.

In the following subsection we provide the construction of  $\chi$  and show that it is a covariance function by means of Euclid's scale mixtures representation and Theorem 4.1. To this end,  $\varphi_{ij}$  is required to be twice differentiable on  $(0, \infty)$  in case  $\varphi$  shall belong to  $\Phi_1^2$  or three times differentiable if  $\varphi$  shall belong to  $\Phi_3^2$ . Constructions in [20] do not allow these degrees of differentiability and we need to extend the approach for the univariate case first.

### 5.1. Cut-off Embedding in the Univariate Case

The function  $\chi$  on  $[0, \infty)$  is defined by

$$\chi(r) = \begin{cases} C + \varphi(r), & 0 \leq r \leq 1, \\ \sum_{i=1}^3 a_i (R-r)^{n_i}, & 1 \leq r \leq R, \\ 0, & r \geq R, \end{cases} \quad (5.1)$$

where  $n_i \geq 4$  and  $n_i \in \mathbb{N}$ ,  $i = 1, 2, 3$ ,  $n_i \neq n_j$ ,  $i \neq j$ , and  $R$  is the solution of the cubic equation

$$\begin{aligned} & \varphi^{(iv)}(1)(R-1)^3 + \varphi'''(1) \left( \sum_{i=1}^3 n_i - 6 \right) (R-1)^2 \\ & + \varphi''(1) \left( \sum_{i \neq j}^3 n_i n_j - 3 \sum_{i=1}^3 n_i + 7 \right) (R-1) \\ & + \varphi'(1) \prod_{i=1}^3 (n_i - 1) = 0. \end{aligned} \quad (5.2)$$

Coefficients  $a_i$ ,  $i = 1, 2, 3$ , and constant  $C$  are

$$\begin{aligned} a_i &= -\frac{\varphi'''(1)(R-1)^2 + \varphi''(1)(n_j + n_k - 3)(R-1) + \varphi'(1)(n_j - 1)(n_k - 1)}{n_i(n_j - n_i)(n_k - n_i)(R-1)^{n_i-1}}, \\ C &= -\varphi(1) + \sum_{i=1}^3 a_i (R-1)^{n_i}, \end{aligned}$$

where  $i \neq j \neq k$ ,  $i, j, k = 1, 2, 3$ . We define the function  $f_k(r)$ ,  $r \in \mathbb{R}$ ,  $k \in \mathbb{N}$  as

$$\begin{aligned} f_k(r) &= k [(k-2)(k-3)(k-5)(k-7)r^4 + (k-3)(7k^2 - 69k + 158)Rr^3 \\ & + 24(k-4)(k-5)R^2r^2 + 48(k-5)R^3r + 48R^4] (R-r)^{k-5}. \end{aligned}$$

**Theorem 5.1.** *Let  $\varphi$  be a continuous function on  $[0, 1]$  such that*

$$-r^{-2}(2\varphi'(r^{1/2}) - 2r^{1/2}\varphi''(r^{1/2}) + r\varphi'''(r^{1/2})) \text{ is convex} \quad (5.3)$$

or

$$48(\varphi''(r)r - \varphi'(r)) - 24r^2\varphi'''(r) + 7r^3\varphi^{(iv)}(r) - r^4\varphi^{(v)}(r) \geq 0 \quad (5.3^*)$$

Furthermore, assume that equation (5.2) has a root  $R \geq 1$ . Then the function  $\chi$  belongs to the class  $\Phi_3$ , if  $C > -1$  and for  $r \in [1, R]$

$$\sum_{i=1}^3 a_i f_{n_i}(r) \geq 0. \quad (5.4)$$

*Proof.* We aim to show that the function  $\chi(r)$  satisfies the third condition in Table 4 in [20], which works also in  $\mathbb{R}^3$  (see again Theorem 1.1. in [18] with  $k = 1$ ,  $l = 1$  and  $\alpha = 1/2$ ),

$$-r^{-2}(2\chi'(r^{1/2}) - 2r^{1/2}\chi''(r^{1/2}) + r\chi'''(r^{1/2})) \text{ is convex.}$$

Setting the polynomial in equation (5.1) and its derivatives up to the order 4 at the point  $r = 1$  to be equal to  $C + \varphi(1)$ ,  $\varphi'(1)$ ,  $\varphi''(1)$ ,  $\varphi'''(1)$ ,  $\varphi^{(iv)}(1)$  respectively, we obtain a system of equations

$$\begin{cases} \sum_{i=1}^3 a_i (R-1)^{n_i} & = C + \varphi(1), \\ \sum_{i=1}^3 a_i n_i (R-1)^{n_i-1} & = \varphi'(1), \\ \sum_{i=1}^3 a_i n_i (n_i-1) (R-1)^{n_i-2} & = \varphi''(1), \\ \sum_{i=1}^3 a_i n_i (n_i-1)(n_i-2) (R-1)^{n_i-3} & = \varphi'''(1), \\ \sum_{i=1}^3 a_i n_i (n_i-1)(n_i-2)(n_i-3) (R-1)^{n_i-4} & = \varphi^{(iv)}(1). \end{cases} \quad (5.5)$$

The system of equations (5.5) reduces to the cubic equation (5.2) with respect to  $R - 1$ . The convexity condition holds true for the polynomial part because of inequality (5.4) and is not violated at the point  $r = 1$  because of the last equation in (5.5).  $\square$

*Remark 5.2.* A solution  $R \geq 1$  of equation (5.2) exists for any  $n_1, n_2, n_3 \geq 4$ , when

$$\varphi^{(iv)}(1)\varphi'(1) < 0.$$

This comes simply from the value of the polynomial at location  $R = 1$  and its behaviour at infinity.

*Remark 5.3.* Assume, that the field  $Y_x$  has a covariance function  $\chi(r)$  and  $C < 0$ . Let  $X$  be a random variable independent of  $Y_x$  and  $X \sim \mathcal{N}(0, -C)$ . Then the field  $Y_x + X$  has the covariance  $\varphi(r)$  on  $G$ .

*Example 5.1* (Univariate powered exponential model). For smaller values of the scale  $a$  the values of  $\alpha$  and  $R$  are positively related (Figure 3). For bigger values of  $a$  this dependence is inverse. This plays a key role in cut-off embedding in the bivariate case. On the one hand a bivariate model given by (4.4) requires  $\beta \geq \frac{\alpha+\gamma}{2}$ , which implies either  $\beta \geq \alpha$  or  $\beta \geq \gamma$  or both, which in turn implies that for smaller values of the scale parameters  $R_{12} > R_{11}$  or  $R_{12} > R_{22}$  respectively. On the other hand in Theorem 5.2 we have the condition  $R_{12} \leq \min\{R_{11}, R_{22}\}$ . This contradiction explains why for small values of the scale parameters the cut-off embedding fails.

*Example 5.2* (Univariate Whittle-Matérn model). Whittle-Matérn covariance model is defined as  $\varphi(r) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} (ar)^\alpha K_\alpha(ar)$ ,  $r \geq 0$ ,  $a, \alpha > 0$ , where  $K_\alpha$  is a modified Bessel function of index  $\alpha$ . We consider  $0 < \alpha \leq 1.5$ , since for higher  $\alpha$  the numerical analysis show that the condition (5.3) is not fulfilled. Figure 4 displays the radius  $R$  depending on the scale  $a$  and the index  $\alpha$ . The dark blue area in Figure 4 signifies that for the given  $n_i$  equation (5.2) does not have a solution  $R > 1$  or inequality (5.4) does not hold true and the construction given by equation (5.6) is not available.

*Example 5.3* (Delay model). Let  $Z_1$  be a weakly stationary isotropic Gaussian random field in  $\mathbb{R}^n$  with covariance function  $C_0 : \mathbb{R}^n \mapsto \mathbb{R}$ . We define  $Z_2(x) = aZ_1(x+h) + \varepsilon(x)$ ,  $x, h \in \mathbb{R}^n$ ,  $a \in \mathbb{R}$ ,  $\varepsilon$  is a weakly stationary isotropic Gaussian random field with a covariance  $C_\varepsilon$ , independent of  $Z_1$ . If  $h$  is a multiple of the grid step size and  $C_0$  has a non-compact support, we can simulate the bivariate field  $Z = (Z_1, Z_2)$  in a rectangular grid  $G = [0, 1/\sqrt{n}]^n$  by means of the cut-off embedding. To do so we simulate a field  $Z_1$  on a bigger grid cutting the covariance function  $C_0$  at the point  $\sqrt{\sum_{i=1}^n (\max\{1/\sqrt{n} + h_i, |h_i|\})^2}$  and take the corresponding part of the simulated  $Z_1$  for  $Z_2$  adding an independently simulated  $\varepsilon(x)$ . Closed formulae for  $R$  and  $\psi$  for some classes of covariance functions are available in [20].

Now we consider the cut-off embedding technique for bivariate fields. We distinguish two cases: one- and three-dimensional fields. Clearly, any simulation method described for three-dimensional fields works also for two-dimensional ones.

## 5.2. Bivariate Fields in $\mathbb{R}$

Consider a bivariate one-dimensional field with the twice differentiable matrix-valued covariance function  $\chi$  defined analogously to the univariate case in the previous section:

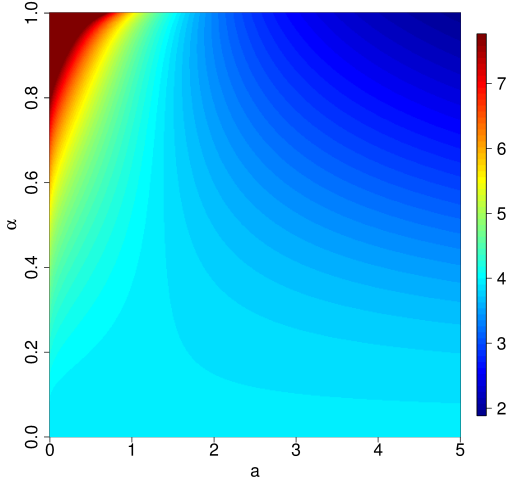


Fig 3: Radius  $R$  for the powered exponential covariance,  $n_1 = 5, n_2 = 6, n_3 = 7$ .

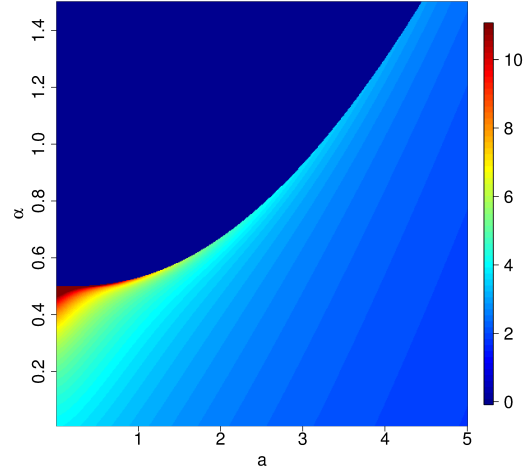


Fig 4: Radius  $R$  for the Whittle-Matern covariance,  $n_1 = 5, n_2 = 6, n_3 = 7$ .

$$\chi_{ij}(r) = \begin{cases} C_{ij} + \varphi_{ij}(r), & r \leq 1, \\ \sum_{k=1}^3 a_{ij,k} (R_{ij} - r)^{n_{ij,k}}, & 1 \leq r \leq R_{ij}, \\ 0, & r \geq R_{ij}. \end{cases} \quad (5.6)$$

The second derivative of  $\chi$  is

$$\chi''_{ij}(r) = \begin{cases} \varphi''_{ij}(r), & r \leq 1, \\ g_{ij}(r), & 1 \leq r \leq R_{ij}, \\ 0, & r \geq R_{ij}, \end{cases}$$

where

$$g_{ij}(r) = \sum_{k=1}^3 a_{ij,k} n_{ij,k} (n_{ij,k} - 1) (R_{ij} - r)^{n_{ij,k}-2}.$$

Assume first, that  $R_{11} \leq R_{12} \leq R_{22}$ . Then it follows

$$\chi''_{11}(r)\chi''_{22}(r) - \rho^2 \chi''_{12}{}^2(r) = -\rho^2 g_{12}^2(r), \quad \text{if } R_{11} \leq r \leq R_{12}. \quad (5.7)$$

This means that the matrix  $\chi''(r)$  is not positive definite for all  $r \geq 0$  and we cannot use Theorem 4.1 to show that  $\chi$  itself is positive definite. The same problem occurs for all  $R_{12} \geq \min\{R_{11}, R_{22}\}$ .

If  $R_{12} \leq \min\{R_{11}, R_{22}\}$ , then

$$\chi''_{11}(r)\chi''_{22}(r) - \rho^2 \chi''_{12}{}^2(r) = \begin{cases} \varphi''_{11}(r)\varphi''_{22}(r) - \rho^2 (\varphi''_{12}(r))^2, & r \leq 1, \\ g_{11}(r)g_{22}(r) - \rho^2 g_{12}^2(r), & 1 \leq r \leq R_{12}, \\ g_{11}(r)g_{22}(r), & 1 \leq r \leq \min\{R_{11}, R_{22}\}, \\ 0, & r \geq \min\{R_{11}, R_{22}\}. \end{cases} \quad (5.8)$$

For the positive definiteness of  $\chi''$  we need all parts of the right hand-side of equation (5.8) to be nonnegative. The first line is nonnegative, if equation (4.1) defines a covariance model and the  $\varphi_{ij}$  have a scale-mixture representation with absolutely continuous distributions. The remaining two lines can be checked only numerically.

Thus we can formulate the following theorem.

**Theorem 5.2.** Let  $\varphi_{ij}(r)$ ,  $i, j = 1, 2$  be the scale mixtures of Euclid's hat  $h_1$  and satisfy the conditions of Theorem 5.1. Then the matrix-valued covariance function  $[\chi_{ij}(r)]_{i,j=1}^2$ , where the  $\chi_{ij}$  are defined via equation (5.6), is positive definite in  $\mathbb{R}$ , if

$$R_{12} \leq \min\{R_{11}, R_{12}\}$$

and if the function (5.8) is nonnegative for all  $r \geq 0$ .

*Remark 5.4.* Analogously to Remark 5.3, for a bivariate field  $Z$  with the covariance matrix  $[\chi_{ij}(r)]_{i,j=1}^2$  and  $C_{12}^2 \leq C_{11}C_{22}$ , the additional random variables are  $X_1 \sim \mathcal{N}(0, -C_{11})$ ,  $X_2 \sim \mathcal{N}(0, -C_{22})$ ,  $X_1$  and  $X_2$  are independent of  $Y_1(x)$  and  $Y_2(x)$  and  $\mathbb{E}[X_1X_2] = -C_{12}$ . Then  $\mathbb{E}[(Z_1(x) + X_1)(Z_2(0) + X_2)] = \mathbb{E}[Z_1(x)Z_2(0)] + \mathbb{E}[X_1X_2] = C_{12} + \varphi_{12}(r) - C_{12} = \varphi_{12}(r)$ , where  $r = \|x\|$ .

### 5.3. Bivariate Fields in $\mathbb{R}^3$

The three-dimensional case is analogous to the one-dimensional one. The only difference is that instead of the matrix  $\chi''(r)$  we consider  $\frac{1}{r}\chi''(r) - \chi'''(r)$  with the following entries

$$\frac{1}{r}\chi''_{ij}(r) - \chi'''_{ij}(r) = \begin{cases} \frac{1}{r}\varphi''_{ij}(r) - \varphi'''_{ij}(r), & 0 \leq r \leq 1, \\ g_{ij}(r)\frac{1}{r} - g'_{ij}(r) & 1 \leq r \leq R_{ij}, \\ 0, & r \geq R_{ij}. \end{cases}$$

**Theorem 5.3.** Let  $\varphi_{ij}(r)$ ,  $i, j = 1, 2$  be the scale mixtures of Euclid's hat  $h_3$  and satisfy the conditions of Theorem 5.1. Then the matrix-valued covariance function  $[\chi_{ij}(r)]_{i,j=1}^2$ , where  $\chi_{ij}$  are defined via equation (5.6) is positive definite in  $\mathbb{R}^3$ , if

$$R_{12} \leq \min\{R_{11}, R_{12}\}$$

and it holds that

$$\begin{aligned} & 0 \leq \left(\frac{1}{r}\chi''_{11}(r) - \chi'''_{11}(r)\right)\left(\frac{1}{r}\chi''_{22}(r) - \chi'''_{22}(r)\right) - \rho^2\left(\frac{1}{r}\chi''_{12}(r) - \chi'''_{12}(r)\right)^2 \\ &= \begin{cases} \left(\frac{1}{r}\varphi''_{11}(r) - \varphi'''_{11}(r)\right)\left(\frac{1}{r}\varphi''_{22}(r) - \varphi'''_{22}(r)\right) - \rho^2\left(\frac{1}{r}\varphi''_{12}(r) - \varphi'''_{12}(r)\right)^2, & 0 \leq r \leq 1, \\ (g_{11}(r)\frac{1}{r} - g'_{11}(r))(g_{22}(r)\frac{1}{r} - g'_{22}(r)) - \rho^2(g_{12}(r)\frac{1}{r} - g'_{12}(r))^2 & 1 \leq r \leq R_{12}, \\ (g_{11}(r)\frac{1}{r} - g'_{11}(r))(g_{22}(r)\frac{1}{r} - g'_{22}(r)) & R_{12} \leq r \leq \min\{R_{11}, R_{22}\}, \\ 0, & r \geq \min\{R_{11}, R_{22}\}. \end{cases} \end{aligned} \quad (5.9)$$

Figures 5, 6 and 7 show simulations from the bivariate powered exponential, bivariate Cauchy and bivariate Matérn models correspondingly. For all simulations we set  $n_{11,1} = n_{12,1} = n_{22,1} = 5$ ,  $n_{11,2} = n_{12,2} = n_{22,2} = 6$ ,  $n_{11,3} = n_{12,3} = n_{22,3} = 7$ . The simulations were performed in R with the RandomFields package [29].

## 6. Acknowledgments

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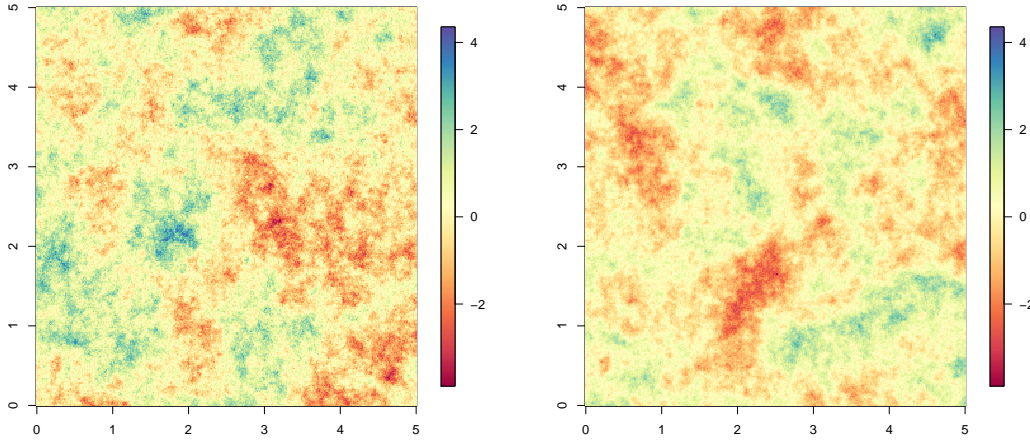


Fig 5: Simulations from the bivariate powered exponential model with variance parameters  $\sigma_{11} = \sigma_{12} = \sigma_{22} = 1$ , scale parameters  $a = 3, b = 4, c = 2$ , smoothness parameters  $\alpha = 0.7, \beta = 0.8, \gamma = 0.9$  and correlation coefficient  $\rho = 0.5$ .

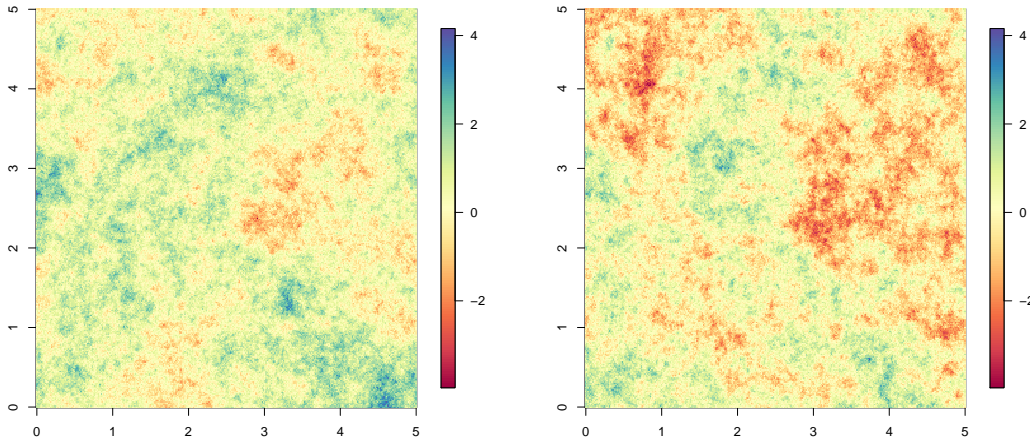


Fig 6: Simulations from the bivariate Cauchy model with variance parameters  $\sigma_{11} = \sigma_{12} = \sigma_{22} = 1$ , scale parameters  $a = 3, b = 3, c = 2.5$ , smoothness parameters  $\alpha = 0.6, \beta = 0.75, \gamma = 7$ , long range parameters  $\lambda = 0.5, \nu = 1, \mu = 1$ , and correlation coefficient  $\rho = 0.5$ .

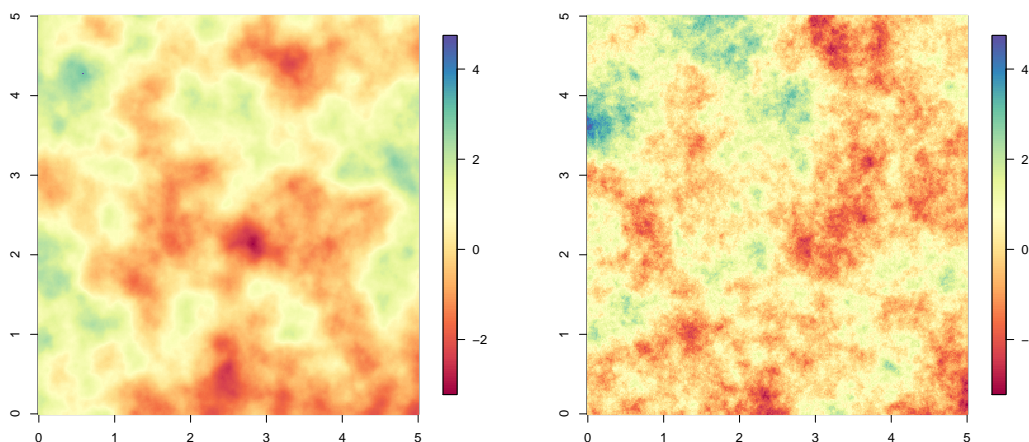


Fig 7: Simulations from the bivariate Matérn model with variance parameters  $\sigma_{11} = \sigma_{12} = \sigma_{22} = 1$ , scale parameters  $a = 3$ ,  $b = 3$ ,  $c = 2$ , smoothness parameters  $\alpha_1 = 1.3$ ,  $\alpha_{12} = 1$ ,  $\alpha_2 = 0.5$  and correlation coefficient  $\rho = 0.4$ .

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